Topics in Applied Stochastic Processes - Midterm 2017/18

Attempt all questions. **Time: 2 hours**

1. Let $\xi_1, \xi_2 \cdots$ be independent and identically distributed real valued random variables with $\phi(\theta) := \mathbb{E}[e^{\theta \xi_1}] < \infty$ for all $\theta \in \mathbb{R}$. Define $\psi(\theta) := \ln \phi(\theta)$.

   (a) Let $S_n = \xi_1 + \xi_2 + \cdots \xi_n$. Show that for each fixed θ, the sequence $X_n = \exp\left[\theta S_n - n\psi(\theta)\right]$ is a martingale with respect to $\{\xi_1, \xi_2, \cdots, \xi_n\}$.   **[5 marks]**

   (b) Show X_n converges with probability 1 to a nonnegative random variable X.   **[2 marks]**

   (c) Assume that $\psi(\theta)$ is strictly convex. Show that $\mathbb{E}\sqrt{X_n} \to 0$ as $n \to \infty$ and conclude that $X = 0$. *(Hint: You might need to use Fatou's Lemma: $\mathbb{E}[\liminf_{n\to\infty} Y_n] \leq \liminf_{n\to\infty} \mathbb{E}[Y_n]$ for a sequence of nonnegative random variables Y_n.)*   **[5 marks]**

2. Consider a *Binomial Asset Pricing Model* with two assets, a bond and a stock. The rate of interest of the bond is 50%. The initial stock price is £100 and the possible rates of interest of the stock are -50% and 100%, both of which are equally likely. We are interested in pricing a European call option with strike price £200, that is $X = (S_2 - 200)_+$, where S_2 is the stock price at time 2.

   (a) Find the arbitrage-free price process for X.   **[6 marks]**

   (b) Find a replicating strategy for X.   **[6 marks]**

3. Suppose that a society adopts the following policy. Allow each couple to have two children. If both children are girls, then allow them to have a third child. Assume (1) the family names are carried only by the male offsprings, (2) everyone gets married and will have 2 or 3 offspring based on the above policy, (3) the probabilities of an offspring being male or female are equal, and (4) sexes of different offsprings are determined independently.

   Suppose there are 3 male members in a family now. What is the probability that the family name survives?   **[6 marks]**